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A Presentation on Explicit Normal Modes for a Perturbed Ocean Model By Perturbation

A Paper Presented at the First IMACS Symposium on Computational Acoustics, 6-8 August 1986, New Haven, Connecticut

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Preface

The work reported in this document was completed under NUSC Project No. A92045, Dr. Rolf Kasper, Principal Investigator. A portion of this research was done while one of us (Gilbert) was on an Intergovernmental Personnel Act Mobility Assignment to the Naval Underwater Systems Center from the University of Delaware. Another portion of this research was completed while two of us (Duston and Verma) participated in the U.S. Navy-ASEE Summer Faculty Research Program at the Naval Underwater Systems Center, New London Laboratory, in conjunction with NUSC Independent Research Project A92045.

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This document is based on a presentation at the First IMACS Symposium on Computational Acoustics, held in New Haven, Connecticut, 6-8 August 1986. (The title was changed from "Effects of Matching Boundary Conditions on Normal Modes Using Perturbation Theory.")

Subject to symplifying assumptions, ~~we are able to~~ use the method of normal modes, involving the eigenvalues and eigenfunctions of a depth dependent Sturm-Liouville problem. ~~We allow~~ the constant sound speed of the idealized uniform ocean model to be modified by a function that is multiplied by a small parameter, epsilon. ~~We~~ investigate the changes that result, using both the classical theory of perturbation for Sturm-Liouville problems and a newer transmutation approach. We are able to use the transmutation approach in a way

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19. ABSTRACT (Cont'd.)

that overcomes its main disadvantage, at least to first order in epsilon. ~~We then obtain~~ explicit formulas for both the eigenvalues and eigenfunctions to first order, without having to expend the eigenfunctions in an infinite series, as is done in the classical approach. Finally, ~~we show that~~ the new formulas are consistent with the classical infinite series.

EXPLICIT NORMAL MODES FOR A PERTURBED OCEAN MODEL BY TRANSMUTATION

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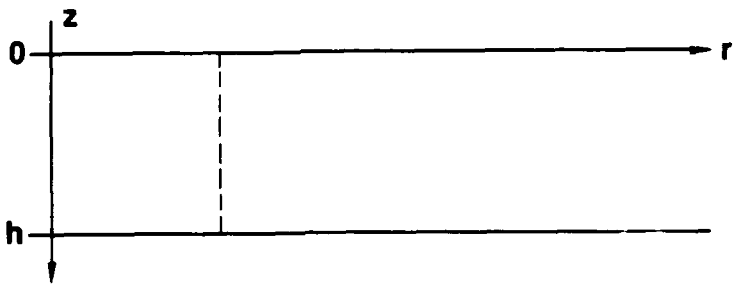
Graph 1

The theoretical foundations of normal mode theory and its use are already well establised. In the future we wish to consider range dependent problems in terms of partitioning, so that the range dependent effects will be corrected at the boundaries of the partitions and the solutions allowed to propagate further. In order to apply this approach it is necessary to find the normal modes at each boundary. We are interested, therefore, in finding a method that will allow us to calculate the normal modes in a faster, more efficient manner. One of the methods that holds promise of attaining this objective is that of transmutation theory. Transmutation theory is essentially a generalization of the method of integral transformation.

THE IDEALIZED OCEAN



$$\left(\begin{array}{l} p_{rr}(r,z) + \frac{1}{r} p_r(r,z) + p_{zz}(r,z) + k^2 p(r,z) = 0, \\ \text{WITH } p(r,0) = 0 \quad \text{AND} \quad p_z(r,h) = 0. \end{array} \right)$$

WE SOLVE USING SEPARATION OF VARIABLES,

$$p(r,z) = \phi(z)\,\theta(r).$$

Graph 2

We are dealing at this point with a fairly simplified model of the ocean and make the following assumptions:

1) uniform depth,
2) ocean is an isotropic medium,
3) sound speed a constant
4) pressure p=0 at the surface, and
5) ocean and its bottom are eventually (at some depth h) underlaid by a rigid surface.

Under these assumptions the excess pressure p satisfies the Helmholtz Equation.

$$p_{rr}(r,z) + \frac{1}{r} p_r(r,z) + p_{zz}(r,z) + k^2 p(r,z) = 0,$$

with the boundary conditions

$$p(r,0) = 0 \qquad \text{and} \qquad p_z(r,h) = 0.$$

The second boundary condition (the boundary condition at h) is that of a rigid subbottom. We solve in a standard manner by using the method of separation of variables $p(r,z) = \psi(z)\Theta(r)$.

THE IDEALIZED STURM-LIOUVILLE PROBLEM

THE IDEALIZED NORMAL MODES SATISFY

$$\phi''(z) + k^2 \phi(z) = \ell\phi(z),$$

$$\phi(0) = 0 \quad \text{AND} \quad \phi'(h) = 0$$

WITH L_2 NORMALIZATION THIS HAS SOLUTIONS

$$\phi_m(z) = \sqrt{\frac{h}{2}} \sin \frac{(2m-1)\pi z}{2h}$$

$$m = 1, 2, 3, \ldots$$

$$\ell_m = k^2 - \left[\frac{(2m-1)\pi}{2h}\right]^2$$

Graph 3

It is well known that for the idealized ocean with constant sound speed the normal modes satisfy an idealized Sturm-Liouville problem given by

$$\phi''(z) + k^2\phi(z) = \ell\phi(z) \ ,$$

and boundary conditions

$$\phi(0) = 0 \qquad \text{and} \qquad \phi'(h) = 0.$$

It is also well known that this problem has a complete set of eigenfunctions given by

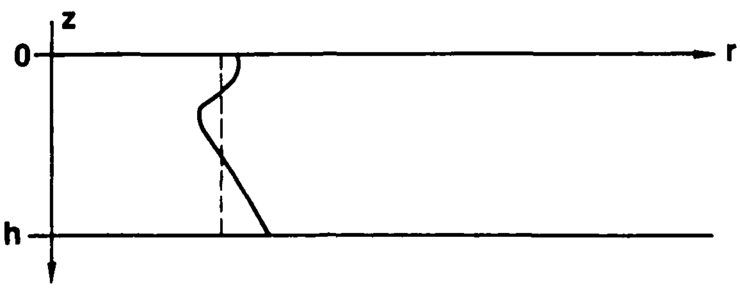
$$\phi_m(z) = \sqrt{\frac{h}{2}} \ \sin\frac{(2m-1)\pi z}{2h} \qquad m = 1, 2, \ldots$$

and corresponding discrete eigenvalues given by

$$\ell_m = k^2 - \left[\frac{(2m-1)\pi}{2h}\right]^2 \qquad m = 1, 2, \ldots .$$

These eigenfunctions have been L_2 normalized.

THE PERTURBED STURM-LIOUVILLE PROBLEM



IN AN OCEAN WHERE THE SOUND SPEED IS A FUNCTION OF DEPTH ONLY THE NORMAL MODES SATISFY

$$\psi''(z) + k^2 n^2(z)\,\psi(z) = \lambda\psi$$

WITH THE CONDITIONS

$$\psi(0) = 0 \text{ AND } \psi'(h) = 0.$$

Graph 4

We wish to solve a similar problem associated with amore general ocean model. The condition that the sound speed be constant is replaced with the assumption that the sound speed is a function of the depth only, the other assumptions remain unchanged. We consider this model to be the <u>perturbed</u> ocean model. The depth dependent equation gives a perturbed Sturm-Liouville problem for the normal modes which satisfy

$$\psi''(z) + k^2 n^2(z)\psi(z) = \lambda\psi(z)$$

with the boundary conditions expressed as

$$\psi(0) = 0 \qquad \text{and} \qquad \psi'(h) = 0.$$

THE INDEX OF REFRACTION

$$n^2(z) = 1 + \epsilon s(z)$$

CONTAINS A <u>PERTURBATION</u> $\epsilon s(z)$.

THE CASE $\epsilon = 0$ REPRESENTS AN <u>IDEALIZED</u> OCEAN WITH CONSTANT SOUND SPEED.

Graph 5

The quantity $n^2(z)$ is the index of refraction and we represent it as

$$n^2(z) = 1 + \epsilon s(z)$$

where the quantity $\epsilon s(z)$ is considered the perturbation. We introduce the parameter ϵ which reflects the size of the perturbation from its average value. We assume that the function $s(z)$ is known for the case of interest. When $\epsilon = 0$ we have recovered the <u>idealized</u> ocean with constant sound speed.

THE GENERAL PERTURBATION APPROACH

FIND THE EIGENFUNCTIONS ψ_n AND THE EIGENVALUES λ_n OF THE PERTURBED PROBLEM IN TERMS OF THE EIGENFUNCTIONS ϕ_n AND EIGENVALUES ℓ_n OF THE IDEALIZED PROBLEM AND THE PERTURBATION $\epsilon s(z)$.

Graph 6

In a perturbation approach we find the eigenfunctions ψ_m and eigenvalues λ_m of the perturbed (depth dependent) in terms of the eigenfunctions ϕ_m and eigenvalues ℓ_m of an idealized problem (which we know) and the perturbation $\epsilon s(z)$ (which we also know). Specifically we look for the changes or corrections which must be made to the idealized eigenfunctions and eigenvalues.

THE CLASSICAL APPROACH

EXPAND IN POWER SERIES IN ϵ

$$\lambda_m = \ell_m + \epsilon\lambda_m^{(1)} + \epsilon^2\lambda_m^{(2)} + \ldots$$

AND

$$\psi_m(z) = \phi_m(z) + \epsilon\psi_m^{(1)}(z) + \epsilon^2\,\psi_m^{(2)}(z) + \ldots,$$

BUT OBTAIN EACH TERM OF $\psi_m(z)$ ONLY AS A FOURIER SERIES

$$\psi_m^{(i)}(z) = \sum_{p=1}^{\infty} \alpha_{mp}^{(i)}\,\phi_p(z).$$

Graph 7

The first approach we examine is the classical perturbation approach found in Titchmarsh. The eigenvalues and eigenfunctions of the perturbed problem are expended in power series of the parameter ϵ.

The perturbed eigenvalue equals the idealized eigenvalue plus corrections,

$$\lambda_m = \ell_m + \epsilon\lambda^{(1)} + \epsilon^2\lambda^{(2)} + \ldots .$$

The perturbed eigenfunctions equals the idealized eigenfunctions plus corrections,

$$\psi_m(z) = \phi_m(z) + \epsilon\psi_m{}^{(1)}(z) + \epsilon^2\psi_m{}^{(2)}(z) + \ldots .$$

However, we find the corrections to the perturbed eigenfunctions only as a Fourier series of the idealized eigenfunctions and we must find the Fourier coefficients α in terms of the idealized eigenfunction, idealized eigenvalues and the perturbation.

THE CLASSICAL RESULTS

THE CORRECTIONS TO FIRST ORDER IN ϵ ARE

$$\lambda_m^{(1)} = \frac{2k^2}{h} \int_0^h s(z) \left[\sin \frac{(2m-1)\pi z}{2h}\right]^2 dz.$$

THE FOURIER COEFFICIENTS ARE

$$\alpha_{mn}^{(1)} = \frac{2k^2}{h} \int_0^h s(z) \sin \frac{(2m-1)\pi z}{2h} \sin \frac{(2n-1)\pi z}{2h} dz,$$

AND

$$\alpha_{mm}^{(1)} = 0.$$

Graph 8

In a fairly straightforward manner the explicit equations for the corrections may be derived. We exhibit the corrections to the first order in ϵ. The first order correction to the m^{th} eigenvalue is given by the formula

$$\lambda^{(1)} = \frac{2k^2}{h} \int_0^h s(z) \left[\sin \frac{(2m-1)\pi z}{2h}\right]^2 dz .$$

The first order correction to the eigenfunctions is given in terms of an infinite fourier series whose coefficients are given by

$$\alpha_{mn}^{(1)} = \frac{2k^2}{h} \int_0^h s(z) \sin \frac{(2m-1)\pi z}{2h} \sin \frac{(2n-1)\pi z}{2h} dz \quad \text{and} \quad \alpha_{mm}^{(1)} = 0.$$

THE TRANSMUTATION APPROACH

WE WANT TO TRANSMUTE SOLUTIONS OF AN IDEALIZED STURM-LIOUVILLE PROBLEM

$$\phi''(z) + k^2\phi(z) = \lambda\phi(z)$$

INTO SOLUTIONS OF THE PERTURBED STURM-LIOUVILLE PROBLEM

$$\psi''(z) + k^2 n^2(z)\psi(z) = \lambda\psi(z).$$

Graph 9

We say we wish to transmute the solutions of an idealized Sturm-Liouville problem into the solutions of the perturbed Sturm-Liouville problem. A transmutation refers to a generalization of the method of integral transformation. It has the advantage that the choice of the idealized system (the system for which the solutions are known) is at our discretion, we have only to find the proper kernel function for the integral transform. As a rule of thumb one chooses the idealized problem to be as similar as possible to the perturbed problem. We choose for our idealized problem a Sturm-Liouville problem with

$$\phi''(z) + k^2\phi(z) = \lambda\phi(z)$$

and wish to transmute the solutions into solutions of the perturbed Sturm-Liouville problem with

$$\psi''(z) + k^2n^2(z)\psi(z) = \lambda\psi(z).$$

THE TRANSMUTATION OPERATION COMPUTES

$$\psi(z) = \varphi(z) + \int_h^z K(z,s)\, \varphi(s)\, ds$$

IN TERMS OF THE TRANSMUTATION KERNEL K(z,s) WHICH MUST SATISFY

$$\frac{\partial^2 K}{\partial z^2} - \frac{\partial^2 K}{\partial s^2} + k^2 (n^2(z) - 1) K = 0.$$

Graph 10

We use a transform of the type given by

$$\psi_m(z) = \phi_n(z) + \int_h^z K(z,s)\phi(s)\, ds$$

where we must specify the kernel function K(z,s).

where the kernel function K(z,s) must be determined. We substitute the transmutation into the separated ordinary differential equation for the variable coefficient Helmholtz equation. This yields a partial differential equation for the kernel function K(z,s)

$$\frac{\partial^2 K}{\partial z^2} - \frac{\partial^2 K}{\partial s^2} + k^2 \epsilon s(z) K = 0 .$$

TO SATISFY THE BOUNDARY CONDITION

$\psi'(h) = 0,$

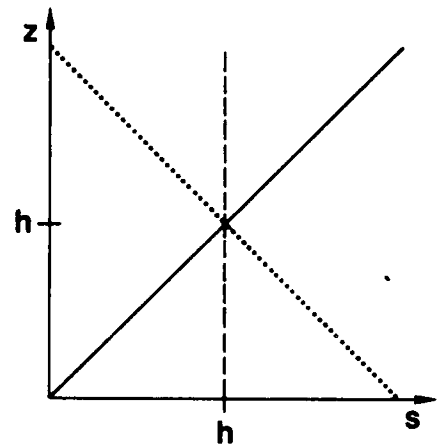
WE ASK THAT THE TRANSMUTATION PRESERVE THE BOUNDARY CONDITION AT THE BOTTOM,

$\phi'(h) = 0$, **WHICH GIVES**

$$\frac{\partial}{\partial s} K(z,h) = 0,$$

AND

$$2 \frac{\partial}{\partial z} K(z,z) + k^2 (n^2(z) - 1) = 0.$$



Graph 11

We now invoke our boundary conditions for the Sturm-Liouville problem with variable coefficients. We ask that $\psi(0)=0$ and $\psi'(h)=0$. We impose the same bottom boundary condition $\phi'(h)=0$ on the solutions of the Sturm-Liouville problem with constant coefficients. Combining this with the given transmutation we obtain two additional constraints on the kernel function $K(z,s)$

$$\frac{\partial}{\partial s} K(z,h) = 0 \qquad \text{and} \qquad 2 \frac{\partial}{\partial z} K(z,z) + k^2 \epsilon s(z) = 0 .$$

These two conditions and the PDE for $K(z,s)$ are sufficient to uniquely determine the kernel function

It would seem that we have made the problem more difficult, we had reduced the problem to an ODE and now we must solve a PDE. Why not directly solve the original Helmholtz equation? The answer is that we will not directly solve the problem but instead will show a way in which the kernel function $K(z,s)$ may be approximated as a power series in the parameter ϵ.

ONE METHOD OF SOLVING FOR THE KERNEL, K(z,s):

LET $M(\xi,\eta) = K(z,s)$ WHERE $2\xi = z + s - 2h$, $2\eta = z - s$.

THEN

$$2\,M_{\xi\eta} + k^2(n^2(\xi + \eta + h) - 1)\,M = 0.$$

EXPAND $M(\xi,\eta)$ AS

$$M(\xi,\eta) = \sum_{p=1}^{\infty} k^{2p} M^{(p)}(\xi,\eta), \text{ WHERE}$$

$$2M^{(1)}(\xi,\eta) = -\int_0^{\xi} (n^2(\alpha+h) - 1)\,d\alpha - \int_0^{\eta} (n^2(\beta+h) - 1)\,d\beta$$

AND

$$M^{(p+1)}(\xi,\eta) = -\int_0^{\eta}\int_0^{\xi} M^{(p)}(\alpha,\beta)\,(n^2(\alpha+\beta+h) - 1)\,d\alpha d\beta.$$

Graph 12

One method of solving for the kernel function $K(z,s) = M(\xi,\eta)$ is to transform into characteristic coordinates given by

$2\xi = z + s - 2h$ and $\qquad 2\eta = z - s$.

We then get the PDE for $M(\xi,\eta)$ which is

$$2\,M_{\xi\eta} + k^2 n^2(\xi + \eta + h)M = 0.$$

We next do a Born expansion of M in powers of the parameter k^2. This yields a sequence of integrals where the first coefficient of the expansion is given by

$$2\,M^{(1)}(\xi,\eta) = -\int_0^{\xi} \epsilon s(\alpha+h)\,d\alpha - \int_0^{\eta} \epsilon s(\beta+h)\,d\beta\ ,$$

and succesive coefficients in the expansion are given by

$$M^{(p+1)}(\xi,\eta) = -\int_0^{\eta}\int_0^{\xi} M^{(p)}(\alpha,\beta)\ \epsilon s(\alpha+\beta+h)\,d\alpha\,d\beta\ .$$

INVERSION OF COORDINATES IN M(ξ,η) GIVES

$$K(z,s) = \sum_{p=1}^{\infty} \epsilon^p K^{(p)}(z,s)$$

$$2\epsilon K^{(1)}(z,s) = -\int_h^{\frac{z+s}{2}} k^2(n^2(\alpha) - 1)\, d\alpha - \int_h^{\frac{z-s+2h}{2}} k^2(n^2(\alpha) - 1)\, d\alpha$$

AND

$$\epsilon K^{(p+1)}(z,s) = -\int_0^{\frac{z-s}{2}} \int_0^{\frac{z+s-2h}{2}} K^p(\alpha,\beta)\, k^2(n^2(\alpha+\beta+h) - 1)\, d\alpha\, d\beta.$$

Graph 13

Inverting coordinates from (ξ,η) back into (z,s) we obtain an expansion of the kernel function $K(z,s)$. The first term of the the expansion is

$$2\epsilon K^{(1)}(z,s) = -k^2 \int_h^{(z+s)/2} \epsilon s(\alpha)\, d\alpha \quad -k^2 \int_h^{(z-s+2h)/2} \epsilon s(\alpha)\, d\alpha$$

and the succesive terms of the expansion are given by

$$\epsilon K^{(p+1)}(z,s) = -k^2 \int_0^{(z-s)/2} \int_0^{(z+s-2h)/2} K^{(p)}(\alpha,\beta)\; \epsilon s(\alpha+\beta+h)\, d\alpha\, d\beta\ .$$

We have now obtained expicit formulas for the power series expansion of the kernel $K(z,s)$ in the parameter ϵ.

TO SATISFY THE BOUNDARY CONDITION AT THE SURFACE $\phi(s)$ IS CONSTRAINED BY

$$0 = \psi(0) = \phi(0) + \int_h^0 K(0,s)\,\phi(s)\,ds.$$

THEREFORE WE DEFINE

$$\theta(z) = \phi(z) - \phi(0).$$

NOTICE THAT

$$\theta''(z) + k^2\,(\theta(z) + \phi(0)) = \lambda(\theta(z) + \phi(0));$$

$$\theta(0) = 0 \text{ AND } \theta'(h) = 0.$$

ITS EXPANSION IS

$$\theta(z) = \theta^{(0)}(z) + \epsilon\theta^{(1)}(z) + \epsilon^2\theta^{(2)}(z) + \cdots.$$

Graph 14

We exhibit the value of the transmuted solution at the surface and see that, in general the transmutation does not preserve the surface boundary condition. Therefore we construct a new function

$\theta(z) = \phi(z) - \phi(0)$

and see that this function satisfies the same boundary conditions as $\psi(z)$. We then expand this new function in power series in ϵ.

SUBSTITUTING THE EXPANSION OF $\theta(z)$ INTO THE STURM-LIOUVILLE PROBLEM GIVES

ORDER ε^0:

$$\theta^{(0)''} + (k^2 - \lambda^{(0)})\,\theta^{(0)} = 0$$

ORDER ε:

$$\theta^{(1)''} + (k^2 - \lambda^{(0)})\,\theta^{(1)} - \lambda^{(1)}\theta^{(0)} = (\lambda^{(0)} - k^2)\,\varphi^{(1)}(0)$$

ORDER ε^2:

$$\theta^{(2)''} + (k^2 + \lambda^{(0)})\,\theta^{(2)} - \lambda^{(1)}\theta^{(1)} - \lambda^{(2)}\theta^{(0)} = (\lambda^{(0)} - k^2)\varphi^{(2)}(0) + \lambda^{(1)}\varphi^{(1)}(0).$$

Graph 15

Substituting $\theta(z)$ into the idealized Sturm-Liouville problem with constant coefficients and isolating the terms with corresponding powers of ϵ we obtain a sequence of ODE's each with the same boundary conditions as the original problem.

For the 0^{th} order term it is well known that this homogeneous ODE has a set of solutions (normalized sine functions) for $\theta^{(0)}$ and corresponding eigenvalues for $\lambda^{(0)}$.

We then take the inner products, $\theta^{(1)}$ with the 0^{th} order equation and $\theta^{(0)}$ with the first order equation subtracting we get the expression

$$\lambda^{(1)}\int_0^h \theta^{(0)}(s)\,ds - \lambda^{(1)} - \int_0^h (\lambda^{(0)} - k^2)\phi^{(1)}(0)\,\theta^{(0)}(s)\,ds \;;$$

which may be solved if we know the value of the constant $\phi^{(1)}(0)$.

THE TRANSMUTATION RESULTS

THE CORRECTIONS TO FIRST ORDER IN ϵ ARE

$$\lambda_n = \lambda_n^{(0)} + \epsilon\left[\frac{(2n-1)\pi}{2h}\sqrt{\frac{2}{h}}\,\phi_n^{(1)}(0)\right]$$

AND

$$\psi_n(z) = \theta_n^{(0)}(z) + \epsilon\left[\phi_n^{(1)}(0)\left(\left[1-\frac{z}{h}\right]\cos\frac{(2n-1)\pi z}{2h} - \int_0^h\left[1-\frac{z}{h}\right]\cos\frac{(2n-1)\pi z}{2h}\,\theta_n^{(0)}(z)\,dz\right)\right.$$

$$\left. + \left(\int_h^z K^{(1)}(z,s)\,\theta_n^{(0)}(s)\,ds - \int_0^h\left[\int_h^z K^{(1)}(z,s)\,\theta_n^{(0)}(s)\,ds\right]\theta_n^{(0)}(z)\,dz\right)\right].$$

Graph 16

We can express the perturbed eigenvalues to the first order in ϵ as

$$\lambda_n = \ell_n + \epsilon\left[\frac{(2n-1)\pi}{2h}\sqrt{\frac{2}{h}}\;\phi_n^{(1)}(0)\right]$$

and the perturbed eigenfunctions as

$$\psi_n(z) = \phi_n(z)$$
$$+\ \epsilon\left\{\phi_n^{(1)}(0)\left(\left[1-\frac{z}{h}\right]\cos\frac{(2n-1)\pi z}{2h} - \int_0^h\left[1-\frac{z}{h}\right]\cos\frac{(2n-1)\pi z}{2h}\,\theta_n^{(0)}(z)\,dz\right)\right.$$
$$\left. +\left(\int_h^z K^{(1)}(z,s)\,\theta_n^{(0)}(s)\,ds - \int_0^h\left[\int_h^z K^{(1)}(z,s)\,\theta_n^{(0)}(s)\,ds\right]\theta_n^{(0)}(z)\,dz\right)\right\}.$$

While the correction to the eigenfunction may look complicated it is the <u>whole</u> correction. We have an explicit formula without having to resort to Fourier series expansion.

THE TRANSMUTATION RESULTS (Cont'd)

THE CORRECTIONS TO FIRST ORDER IN ϵ ARE DEFINED IN TERMS OF

$$\phi_n^{(1)}(0) = \int_0^h K^{(1)}(0,s)\,\theta_n^{(0)}(s)\,ds$$

AND

$$\epsilon K^{(1)}(z,s) = -\frac{k^2}{2}\left(\int_h^{\frac{z+s}{2}} [n^2(\zeta) - 1]\,d\zeta + \int_h^{\frac{z-s+2h}{2}} [n^2(\zeta) - 1]\,d\zeta\right).$$

GRAPH 17

The corrections for the eigenvalues and eigenfunctions are expressed in terms of $\phi^{(1)}(0)$ and $K^{(1)}(z,s)$. The first constant is defined by the expression

$$\phi_n^{(1)}(0) = \int_0^h K^{(1)}(0,s)\theta_n^{(0)}(s)\,ds,$$

and the first order in epsilon term of the power series expansion of the kernel is given by

$$\epsilon K^{(1)}(z,s) = -\frac{k^2}{h}\left(\int_h^{(z+s)/2} [n^2(\zeta) - 1]\,d\zeta + \int_h^{(z-s)/2 + h} [n^2(\zeta) - 1]\,d\zeta\right).$$

We again have the perturbation type result where the corrections are explicit functions of θ, the unperturbed eigenfunction, ℓ, the unperturbed eigenvalue and the expression $[n^2(z) - 1]$ which is the perturbation.

CONSISTENCY OF THE TWO APPROACHES

RECALL

$$\epsilon\lambda_n^{(1)} = \epsilon\left[\frac{(2n-1)\pi}{2h}\sqrt{\frac{2}{h}}\,\phi_n^{(1)}(0)\right]$$

$$= \frac{(2n-1)\pi}{2h}\sqrt{\frac{2}{h}}\int_0^h \frac{k^2}{2}\left(\int_h^{\frac{s}{2}}[n^2(\zeta)-1]\,d\zeta + \int_h^{\frac{s}{2}+h}[n^2(\zeta)-1]\,d\zeta\right)\theta_n^{(0)}(s)\,ds.$$

INTEGRATION BY PARTS GIVES

$$\epsilon\lambda_n^{(1)} = \frac{2k^2}{h}\int_0^h (n^2(\zeta)-1)\left[\sin\frac{(2n-1)\pi\xi}{2h}\right]^2 d\zeta.$$

Graph 18

The results of the two methods look disimilar, however we can show that the results are consistent. To show consistency of the first order correction of the eigenvalues we start with

$$\epsilon\lambda_n^{(1)} = \epsilon\left[\frac{(2n-1)\pi}{2h}\sqrt{\frac{2}{h}}\ \phi_n^{(1)}(0)\right] .$$

We substitute for $\phi^{(1)}(0)$, including the power series expression for the kernel function. Integration by parts yields

$$\epsilon\lambda_n^{(1)} = \frac{2k^2}{h}\int_0^h [n^2(\zeta)-1]\left[\sin\frac{(2n-1)\pi\zeta}{2h}\right]^2 d\zeta ,$$

which is identical to Titchmarsh.

WE CAN SHOW THAT THE FOURIER COEFFICIENTS OF THE FIRST ORDER CORRECTION DEFINED BY

$$\alpha_{nm} = \int_0^h \psi_n^{(1)}(z)\,\theta_m^{(0)}(z)\,dz,$$

ARE GIVEN BY

$$\epsilon\alpha_{nm} = \frac{-2k^2}{h(\lambda_m^{(0)} - \lambda_n^{(0)})} \int_0^h (n^2(z) - 1) \sin\frac{(2n-1)\pi z}{2h} \sin\frac{(2m-1)\pi z}{2h}\,dz$$

AND

$$\epsilon\,\alpha_{nn} = 0.$$

Graph 19

The correction given by the transmutation is radically different from the classical results. However if we express the transmutation correction in Fourier series with respect to the functions $\theta^{(0)}$, we see that the Fourier coefficients

$$\alpha_{nm} = \int_0^h \psi_n^{(1)}(z)\ \theta_m^{(0)}(z)\ dz,$$

result.

After again making substitutions for $\phi^{(1)}(0)$ and $K^{(1)}(z,s)$, several integrations by parts and some changes of variable these coefficients may be used to derive the classical Fourier coeffecients

$$\epsilon\alpha_{nm} = \frac{-2k^2}{h(\lambda_m^{(0)} - \lambda_n^{(0)})} \int_0^h [n^2(z) - 1] \sin\frac{(2n-1)\pi z}{2h} \sin\frac{(2m-1)\pi z}{2h}\ dz$$

and

$$\epsilon\alpha_{nm} = 0.$$

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